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Dynamics and Lax–Phillips scattering for generalized Lamb models

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Abstract

This paper treats the dynamics and scattering of a model of coupled oscillating systems, a finite dimensional one and a wave field on the half line. The coupling is realized producing the family of self-adjoint extensions of the suitably restricted self-adjoint operator describing the uncoupled dynamics. The spectral theory of the family is studied and the associated quadratic forms constructed. The dynamics turns out to be Hamiltonian and the Hamiltonian is described, including the case in which the finite-dimensional systems comprise nonlinear oscillators; in this case, the dynamics is shown to exist as well. In the linear case, the system is equivalent, on a dense subspace, to a wave equation on the half line with higher order boundary conditions, described by a differential polynomial $p(\partial_x)$ explicitly related to the model parameters. In terms of such structure, the Lax–Phillips scattering of the system is studied. In particular, we determine the scattering operator, which turns out to be unitarily equivalent to the multiplication operator given by the rational function $-p(\mathrm{i}\kappa)^*/p(\mathrm{i}\kappa)$, the incoming and outgoing translation representations and the Lax–Phillips semigroup, which describes the evolution of the states which are neither incoming in the past nor outgoing in the future.

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1. Introduction

In this paper, we investigate the spectral theory, dynamics and Lax–Phillips scattering for an abstract system which models the interaction between a finite-dimensional linear subsystem and an infinite-dimensional wave field on a half line. We will call such systems *generalized Lamb models* in that they extend the standard Lamb model (see [10]) that will be introduced shortly. Although our main concern is with linear oscillators, we will describe some properties of the models in the anharmonic case also.

To introduce our models, let us consider an n-dimensional Lagrangian system linearized around a certain equilibrium point. Its equations of motion are given by

$$G\ddot{y} = Hy$$

where $y \in \mathbb{R}^n$ is the displacement from the given equilibrium point (for reference, $y = 0$), and the matrices G and H represent the quadratic approximation of kinetic and potential energy around the equilibrium point. G is positive definite and both matrices are symmetric with respect to the standard inner product in $\mathbb{R}^n$. For technical and theoretical reasons, it is more useful to endow $\mathbb{R}^n$ with the inner product given by G. With respect to this inner product, the matrix $L = G^{-1}H$ is symmetric and the Lagrangian equation takes the form

$$\ddot{y} = Ly$$

with L symmetric with respect to the G inner product. The case of a chain of harmonic oscillators is well known and yields to a Jacobi matrix for the operator L.

Analogously, let us consider the wave equation on the half line. To be definite let us consider Neumann boundary condition at the origin. Denoting with Δ_N the one-dimensional Laplacian with homogeneous Neumann boundary conditions at the origin, the wave field (we have posed equal to one the wave velocity) evolves according to the wave equation

$$\ddot{\phi} = \Delta_N \phi.$$

So we have two decoupled second-order equations for two different oscillating systems, the finite dimensional one with generator L and the infinite dimensional one with generator Δ_N.

Thus, on the direct sum $L^2(\mathbb{R}_+) \oplus \mathbb{R}^n$ we have the self-adjoint operator

$$A_0 = \Delta_N \oplus L$$

and the corresponding abstract wave equation

$$\ddot{\Psi} = A_0 \Psi.$$

In a heuristic way, a coupling between the two oscillating systems could be given by posing a constraint between boundary values of the wave field at the origin and the displacement of the finite-dimensional system. The prototype of this coupling is furnished by the well-known Lamb model where a semi-infinite string is coupled to a single particle oscillating in the transverse direction (see section 4.1 for the general case of a chain of oscillators); the particle, with mass M, is subjected to the tension T of the string at the origin and to a restoring harmonic force with spring constant K, so that the formal equations are given by the system

$$\ddot{\phi}(t,x) = \phi''(t,x) \qquad x > 0,$$
$$M\ddot{y}(t) = -Ky(t) + T\phi'(t, 0_+),$$

plus the constraint

$$\phi(t, 0_+) = y(t).$$

This model was proposed by Horace Lamb in 1900 as an example of dissipation in (subsystems of) conservative systems. In fact, it is possible to decouple field and particle dynamics, and the particle component satisfies a reduced equation which turns out to be, for $t > 0$,

$$M\ddot{y}(t) + 2T\dot{y}(t) + Ky(t) = T(\phi_0'(t, 0_+) + \dot{\phi}_0(t, 0_+)).$$

The forcing term on the right-hand side depends on the evaluation at the origins of the *free* evolution for the wave field of the initial data $\phi_0, \dot{\phi}_0$. Thus, for initial data of compact support the forcing term is a pure transient definitively vanishing, and the reduced dynamics for the particle coincides for large times with that of a damped harmonic oscillator, so that the effect of interaction between particle and field reduces to damping only. This means exponentially fast return to equilibrium of the finite-dimensional subsystem and correspondingly a neat transfer of energy to the field. This relaxation property towards the equilibrium position of the finite-dimensional component is always true when the corresponding self-adjoint operator has empty point spectrum (see remark 6.1) as it is the case in the Lamb model. The result has various generalizations to anharmonic oscillators (see [8]).

Some tridimensional models reduce themselves to generalized Lamb models due to symmetry. The case of an elastic spherical shell coupled to the acoustic field when only radial oscillations are allowed is treated in section 4.3, and it yields to a nontrivial generalized Lamb model. Another issue of interest of these coupled systems is given by the fact that some linear models of classical and quantum field theory reduce themselves in the ultraviolet limit, and after due renormalizations, to generalized Lamb models. For example, the Schwabl–Thirring (see [16]) model when restricted to its monopole sector (the only one where it is not trivial) and after a spring constant renormalization turns out to be equivalent to a Lamb model (see [12]). A similar phenomenon occurs for the Pauli–Fierz model describing the interaction of a charged oscillator with the electromagnetic field in dipole approximation and after mass renormalization (see [3]). In this case, however, the reduction of the dynamics on its nontrivial part yields a boundary condition different from that of the Lamb model (see example 4.2).

This discussion of motivating examples, and relevant studies existing in the literature, shows however that the usual formulation is partly formal in that it is not clear what should be the functional setting of the Lamb system in the first place, and secondarily its Hamiltonian formulation, if any exists.

A guide to set rigorously these questions in this and in more general situations is suggested by an analysis of the coupling between field and particle. The idea is to restrict the uncoupled vector operator A_0 to the linear variety defined by the constraint existing between field and subsystem; the uncoupled operator on this linear variety is no more self-adjoint but it is symmetric with defect indices $(1, 1)$. All possible self-adjoint extensions different from A_0 itself correspond to a well-defined coupling or interaction between two subsystems. The case of the Lamb model, for example, corresponds to the closed linear variety $\phi(0_+) = y$. The most general boundary condition still producing a self-adjoint operator, as we will see, is of the kind $\theta\phi'(0_+) + \phi(0_+) = w \cdot y$, where $\theta \in \mathbb{R}$ and w is a given vector in $\mathbb{R}^n$. The case of a chain of harmonic oscillators one of which coincides with the boundary point of the string, which is the more obvious generalization of the Lamb model, corresponds to a vector w with just a single entry nonvanishing and to $\theta = 0$. A generic w corresponds to nonlocal coupling between string and more than one oscillator, i.e. the interaction is not 'nearest neighbour'. Our first concern is to give a rigorous account of this construction and to explicitly describe the interacting system so obtained (see theorem 2.1) as well as its spectral properties (see theorem 2.2). The interacting operator so constructed is a singular perturbation of the self-adjoint operator A_0, related to the class of one-dimensional point interactions, or better point interaction with inner structure previously studied in different context and with a different formalism by many authors (see, e.g., [1, 9, 13] and references therein). In passing, we note that the coupled operator we study corresponds to a boundary value problem for the wave field only, but with an eigenparameter-dependent boundary condition (see remark 2.5); this sort of parameter-dependent boundary value problems are well known in the literature, both physical and mathematical. However, we do not follow this road to the study of spectral and scattering properties of the coupled operator.

As a byproduct of the construction, we obtain in section 3 the Hamiltonian structure of the system (see theorem 3.2), which we generalize to the case of anharmonic oscillators, giving conditions for the existence of global flow (see remark 3.3). As far as we know, a completely rigorous description of the Hamiltonian structure of such type of systems has been lacking up to now, whereas interesting, but formal treatments are scattered in the literature (see, for example, [7, 12]).

In the case the symmetric operator L has no degenerate eigenvalues we show that the dynamics of the system is equivalent, for a dense set of smooth initial data, to a reduced dynamics of a wave equation on the half line which incorporates the interaction with the finite-dimensional systems through a higher order boundary condition of the kind $p(\partial_x)\phi(t, 0_+) = 0$, where the polynomial p is explicitly related to the parameters entering into the definition of the model (see theorem 5.1). This is a technical result, useful for the analysis of the Lax–Phillips scattering for the system, which is the main topic of the remaining part of the paper.

In section 6, in the case of empty point spectrum, we determine the incoming (R^-) and outgoing (R^+) translation representations which make the dynamics unitarily equivalent to the translation on $L^2(\mathbb{R})$ defined by $T^t f(x) := f(x-t)$. This provides the scattering operator S_p^* for the system by the relation $S_p^* = R^+(R^-)^{-1}$. Moreover, S_p^* turns out to be unitarily equivalent to the multiplication operator given by the rational function $-p(\mathrm{i}\kappa)^*/p(\mathrm{i}\kappa)$. In section 7, the Lax–Phillips semigroup Z^t, $t \geqslant 0$, which describes the evolution of the states which are neither incoming in the past nor outgoing in the future is completely characterized. It acts on a finite-dimensional vector space, whose dimension coincides with the degree of the polynomial p, by $Z^t = \mathrm{e}^{-tB}$, where the spectrum of the generator B is made of the resonances of the system. Such resonances correspond to the roots of the polynomial p.

2. Singular perturbations of the free dynamics

Let us begin with some definitions. We denote by $L^2(\mathbb{R}_+)$ the Hilbert space of square-integrable functions on the half line $(0, +\infty)$ and by $H^1(\mathbb{R}_+)$ and $H^2(\mathbb{R}_+)$ the Sobolev spaces:

$$H^1(\mathbb{R}_+) := \{\phi \in L^2(\mathbb{R}_+) : \phi' \in L^2(\mathbb{R}_+)\},$$

$$H^2(\mathbb{R}_+) := \{\phi \in L^2(\mathbb{R}_+) : \phi', \phi'' \in L^2(\mathbb{R}_+)\}.$$

Here, the prime ϕ' denotes a spatial derivative. With a dot, $\dot{\phi}$, we will denote a time derivative. We then define $H^2_N(\mathbb{R}_+)$ as the subspace of $H^2(\mathbb{R}_+)$ of functions which satisfy homogeneous Neumann boundary conditions at zero, i.e.,

$$H^2_N(\mathbb{R}_+) := \{\phi \in H^2(\mathbb{R}_+) : \phi'(0_+) = 0\}.$$

We denote by $\langle\cdot,\cdot\rangle_2$ and by $\|\cdot\|_2$ the usual scalar product and the corresponding norm on $L^2(\mathbb{R}_+)$.

Given the n-dimensional Hilbert space $\mathfrak{h}$ with scalar product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$, and given the symmetric operator $L : \mathfrak{h} \to \mathfrak{h}$, we consider the self-adjoint operator

$$A_0 : H^2_N(\mathbb{R}_+) \oplus \mathfrak{h} \subset L^2(\mathbb{R}_+) \oplus \mathfrak{h} \to L^2(\mathbb{R}_+) \oplus \mathfrak{h}, \qquad A_0(\phi, y) := (\phi'', Ly).$$

Regarding the spectrum of A_0 one has

$$\sigma_{\mathrm{ess}}(A_0) = \sigma_{\mathrm{ac}}(A_0) = (-\infty, 0], \qquad \sigma_{\mathrm{pp}}(A_0) = \sigma(L).$$

In order to couple the two dynamical systems described by the equations $\ddot{\phi} = \phi''$ and $\ddot{y} = Ly$, we define the continuous and surjective linear operator

$$\tau : H^1(\mathbb{R}_+) \oplus \mathfrak{h} \to \mathbb{C}, \qquad \tau(\phi, y) := \phi(0_+) - \langle w, y\rangle, \qquad w \in \mathfrak{h},$$

and then we consider the closed symmetric operator $\dot{A}_0$ obtained by restricting A_0 to the kernel of τ. $\dot{A}_0$ has deficiency indices $(1, 1)$ and we are interested in its self-adjoint extensions different from A_0 itself, which we parametrize by the real extension parameter θ. Thus to each quadruple $(\mathfrak{h}, L, w, \theta)$ corresponds a different *generalized Lamb model*. The next theorem completely characterizes such models.

Theorem 2.1. *For any $\theta \in \mathbb{R}$ the linear operator*

$$A : D(A) \subset L^2(\mathbb{R}_+) \oplus \mathfrak{h} \to L^2(\mathbb{R}_+) \oplus \mathfrak{h}$$

defined by

$$D(A) := \{(\phi, y) \in H^2(\mathbb{R}_+) \oplus \mathfrak{h} : \theta\phi'(0_+) + \phi(0_+) = \langle w, y\rangle\},$$

$$A(\phi, y) := (\phi'', Ly + w\phi'(0_+))$$

is a self-adjoint extension of $\dot{A}_0$ and its resolvent is given by

$$(-A + z)^{-1} = (-A_0 + z)^{-1} + (\theta + \Gamma(z))^{-1} G_z \otimes G_{z^*},$$

where

$$\Gamma(z) := -\left(\pm\frac{1}{\sqrt{z}} + \langle w, (-L + z)^{-1} w\rangle\right), \qquad \pm\text{Re}\,\sqrt{z} > 0$$

and

$$G_z = \left(\pm\frac{e^{\mp\sqrt{z}x}}{\sqrt{z}}, -(-L + z)^{-1} w\right), \qquad \pm\text{Re}\,\sqrt{z} > 0.$$

Proof. We will make use of the mathematical procedure developed in [14] (see also [4], theorem 2.2, for a similar proof in the case of a one-dimensional model in acoustics).

For any $z \in \rho(A_0)$, let us consider the two linear continuous operators:

$$\breve{G}(z) : L^2(\mathbb{R}_+) \oplus \mathfrak{h} \to \mathbb{C}, \qquad \breve{G}(z) := \tau(-A_0 + z)^{-1},$$

$$G(z) : \mathbb{C} \to L^2(\mathbb{R}_+) \oplus \mathfrak{h}, \qquad G(z) := \breve{G}(z^*)^*.$$

Since

$$\left(-\frac{d^2}{dx^2} + z\right)^{-1} : L^2(\mathbb{R}_+) \to H^2_N(\mathbb{R}_+)$$

has kernel

$$\mathcal{G}_N(z; x_1, x_2) = \pm\frac{e^{\mp\sqrt{z}|x_1 - x_2|} + e^{\mp\sqrt{z}(x_1 + x_2)}}{2\sqrt{z}}, \qquad \pm\text{Re}\,\sqrt{z} > 0,$$

the operators $\breve{G}(z)$ and $G(z)$ are represented by the vectors G_{z^*} and G_z, respectively, where

$$G_z = \left(\pm\frac{e^{\mp\sqrt{z}x}}{\sqrt{z}}, -(-L + z)^{-1} w\right), \qquad \pm\text{Re}\,\sqrt{z} > 0.$$

Note that

$$\text{Ran}(G(z)) \cap D(A_0) = \{0\}. \tag{2.1}$$

Now we define, for any $z \in \rho(A_0)$, the map

$$\Gamma(z) : \rho(A_0) \to \mathbb{C}, \qquad \Gamma(z) := -\tau G(z),$$

i.e.,

$$\Gamma(z) := -\left(\pm\frac{1}{\sqrt{z}} + \langle w, (-L+z)^{-1}w\rangle\right), \qquad \pm\mathrm{Re}\sqrt{z} > 0.$$

At first, let us note that the function Γ satisfies the relation

$$\Gamma(z) - \Gamma(w) = (z-w)\,\breve{G}(w)G(z). \tag{2.2}$$

Indeed,

$$\Gamma(z) - \Gamma(w) = \tau(G(w) - G(z))$$

and, by first resolvent identity and by the definition of $G(z)$,

$$(z-w)(-A_0+z)^{-1}G(z) = G(w) - G(z).$$

Relation (2.2) implies that

$$R(z) := (-A_0+z)^{-1} + (\theta+\Gamma(z))^{-1}G_z \otimes G_{z^*}$$

satisfies the first resolvent equation

$$R(w) - R(z) = (z-w)R(w)R(z). \tag{2.3}$$

By the definitions of $\breve{G}(z)$ and $G(z)$, and since $\Gamma(z)^* = \Gamma(z^*)$, one obtains

$$R(z)^* = R(z^*). \tag{2.4}$$

Moreover, by (2.1), the linear operator $R(z)$ is injective. Thus,

$$A := -R(z)^{-1} + z$$

is well defined on the domain

$$D(A) := \mathrm{Range}(R(z)).$$

By (2.3) such a definition is z-independent. By (2.4), A is symmetric and is self-adjoint since

$$\mathrm{Range}(-A \pm i) = L^2(\mathbb{R}_+) \oplus \mathfrak{h}$$

by construction. We have thus defined the self-adjoint operator

$$D(A) := \left\{(\phi_z, y_z) + (\theta+\Gamma(z))^{-1}(\phi_z(0_+) - \langle w, y_z\rangle)G_z,\, \phi_z \in H^2_N(\mathbb{R}_+)\right\},$$

$$(-A+z)(\phi, y) := (-A_0+z)(\phi_z, y_z).$$

This implies

$$\phi'(0_+) = -(\theta+\Gamma(z))^{-1}(\phi_z(0_+) - \langle w, y_z\rangle)$$

and

$$\phi(0_+) = \phi_z(0_+) \mp \frac{1}{\sqrt{z}}\phi'(0_+).$$

Therefore,

$$\begin{aligned}\theta\phi'(0_+) &= (\theta+\Gamma(z))\phi'(0_+) - \Gamma(z)\phi'(0_+)\\ &= -\phi_z(0_+) + \langle w, y_z\rangle + \left(\frac{\pm 1}{\sqrt{z}} + \langle w, (-L+z)^{-1}w\rangle\right)\phi'(0_+)\\ &= -\phi'(0_+) + \langle w, (y_z + (-L+z)^{-1}w\rangle\phi'(0_+))\\ &= -\phi(0_+) + \langle w, y\rangle.\end{aligned}$$

Posing

$$A(\phi, y) \equiv (A_1(\phi, y), A_2(\phi, y))$$

one obtains

$$A_1(\phi, y)(x) = \phi_z''(x) \mp z\phi'(0_+)\frac{e^{\mp\sqrt{z}x}}{\sqrt{z}}$$
$$= \left(\phi_z(x) \mp \phi'(0_+)\frac{e^{\mp\sqrt{z}x}}{\sqrt{z}}\right)'' = \phi''(x)$$

and

$$A_2(\phi, y) = Ly_z + z\phi'(0_+)(-L+z)^{-1}w$$
$$= Ly + (-L(-L+z)^{-1} + z(-L+z)^{-1})w\phi'(0_+)$$
$$= Ly + w\phi'(0_+).$$

□

Let us define the (eventually empty) set

$$\sigma_w(L) := \left\{\lambda \in \sigma(L) : w \in \mathfrak{h}_\lambda^\perp\right\},$$

where $\mathfrak{h}_\lambda$ denotes the spectral subspace relative to λ.

Theorem 2.2.

$$\sigma_{\rm ess}(A) = \sigma_{\rm ac}(A) = (-\infty, 0],$$

$$\sigma_{\rm pp}(A) = \sigma_w(L) \cup \{\lambda \in \rho(L) \cap (0, +\infty) : \theta + \Gamma(\lambda) = 0\}.$$

Proof. The properties regarding the essential and continuous spectrum are more or less standard and can be obtained proceeding as in [4], theorem 2.3. Let us now come to the point spectrum.

1. Let $\lambda \in \sigma(L)$. Then $(0, y_\lambda)$ is an eigenvector if y_λ solves the equations

$$Ly_\lambda = \lambda y_\lambda, \qquad \langle w, y_\lambda\rangle = 0,$$

thus $\lambda \in \sigma_w(L)$.

2. Let $\lambda > 0$. Then $\phi_\lambda(x) := e^{-\sqrt{\lambda}x}$ solves $\phi_\lambda'' = \lambda\phi_\lambda$. Thus $(\phi_\lambda, y_\lambda)$ is an eigenvector if λ and y_λ solve the equations

$$Ly_\lambda - \sqrt{\lambda}\, w = \lambda y_\lambda, \qquad -\theta\sqrt{\lambda} + 1 - \langle w, y_\lambda\rangle = 0. \tag{2.5}$$

If $\lambda \in \rho(L)$ then

$$y_\lambda = -\sqrt{\lambda}(-L+\lambda)^{-1}w$$

and λ must solve the equation

$$-\theta\sqrt{\lambda} + 1 + \sqrt{\lambda}\langle w, (-L+\lambda)^{-1}w\rangle = 0.$$

If otherwise $\lambda \in \sigma(L)$ then (2.5) can be solved only if $w \in \mathfrak{h}_\lambda^\perp$ by $y_\lambda = y_\lambda^\parallel + y_\lambda^\perp$, where $y_\lambda^\parallel \in \mathfrak{h}_\lambda$ and $y_\lambda^\perp \in \mathfrak{h}_\lambda^\perp$ are defined by

$$y_\lambda^\perp := -\sqrt{\lambda}(-L_\lambda+\lambda)^{-1}w, \qquad L_\lambda := (1-P_\lambda)L(1-P_\lambda) : \mathfrak{h}_\lambda^\perp \to \mathfrak{h}_\lambda^\perp.$$

Thus $\lambda \in \sigma_w(L)$ and moreover it has to solve the equation

$$-\theta\sqrt{\lambda} + 1 + \sqrt{\lambda}\langle w, (-L_\lambda+\lambda)^{-1}w\rangle = 0.$$

□

Remark 2.3. When $\sigma_w(L)$ is empty, i.e. in the generic situation, the point spectrum of the interacting operator A is quite different from the point spectrum of the decoupled one, A_0. In particular, the free eigenvalues of the finite-dimensional subsystem disappear, and in their place the real solutions of the equation $\Gamma(\lambda) + \theta = 0$ could possibly appear. In fact, as we shall see in section 6, the disappeared eigenvalues, which for the noninteracting operator A_0 are immersed in the continuum spectrum, become resonances of the interacting operator.

Remark 2.4. In the case $\sigma_w(L) \neq \emptyset$ we can suppose, without loss of generality, $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, $L = L_1 \oplus L_2$ and $w = w_1 \oplus 0$. Then for the self-adjoint extensions given in theorem 2.1 we have $A = A_1 \oplus L_2$, where

$$A_1 : D(A_1) \subset L^2(\mathbb{R}_+) \oplus \mathfrak{h}_1 \to L^2(\mathbb{R}_+) \oplus \mathfrak{h}_1$$

is defined by

$$D(A_1) := \{(\phi, y_1) \in H^2(\mathbb{R}_+) \oplus \mathfrak{h}_1 : \theta\phi'(0_+) + \phi(0_+) = \langle w_1, y_1\rangle_{\mathfrak{h}_1}\},$$

$$A_1(\phi, y_1) := (\phi'', L_1 y_1 + w_1\phi'(0_+)),$$

i.e., the dynamics on $\mathfrak{h}_2$ is trivial, in the sense that it is decoupled from the field one. Thus, the hypothesis $\sigma_w(L) \neq \emptyset$ is equivalent to the nonexistence of a subspace on which the particles and the field are uncoupled. In other words, the points of the pure point spectrum belonging to $\sigma_w(L)$ correspond to 'radiationless motions', in which the interaction between oscillators and field is decoupled. Similar exceptional solutions in which the subsystem oscillates at a normal frequency of the decoupled system are known, for example, also in the classical electrodynamics of an extended charge where they are called Bohm–Weinstein modes. In that case, the coupling between field and particle is defined by the charge density $\rho(x)$ of the particle, and the condition to have radiationless modes of frequency ω is that the Fourier transform of the form factor satisfies $\hat{\rho}(\omega) = 0$.

Remark 2.5. The operator A can be interpreted in a formal way as a differential operator with an eigenvalue-dependent boundary condition. Let us consider the secular equation for the operator A and in particular its finite-dimensional component, and couple it with the boundary condition for elements of the domain of the operator. We get

$$Ly + w\phi'(0_+) = \lambda y \qquad \theta\phi'(0_+) + \phi(0_+) = \langle w, y\rangle.$$

From the first equation it follows $y = -\phi'(0_+)(L - \lambda)^{-1}w$ and substituting into the second equation one gets

$$(\theta + \langle w, (L - \lambda)^{-1}w\rangle)\phi'(0_+) + \phi(0_+) = 0$$

which is, formally, a Robin boundary condition for the field at the origin. The condition contains the eigenvalue λ and it is known in the physical and mathematical literature as an energy-dependent boundary condition. From this point of view, the boundary value problem for the coupled operator can be reduced to a boundary value problem for the field only, but eigenvalue dependent.

By the way note that, as it should be, the above eigenvalue-dependent boundary condition is equivalent to the eigenvalue equation in theorem 2.3, $\Gamma(\lambda) + \theta = 0$, as it is immediately seen by the position $\sqrt{z} = \lambda$.

3. The Hamiltonian structure

In this section, we are interested in describing the Hamiltonian structure of the dynamical system related to the abstract wave equation:

$$\ddot{\Psi} = A\Psi, \qquad \Psi \equiv (\phi, y).$$

The solution of the corresponding Cauchy problem is then given by the symplectic flow generated by the linear operator $\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix}$.

We begin by determining the quadratic form corresponding to $-A$:

Theorem 3.1. *Let us denote by Q the quadratic form of $-A$.*

1. If $\theta = 0$ then

$$D(Q) = \{(\phi, y) \in H^1(\mathbb{R}_+) \oplus \mathfrak{h} : \phi(0_+) = \langle w, y \rangle\},$$

$$Q : D(Q) \to \mathbb{R}, \qquad Q(\phi, y) = \|\phi'\|_2^2 - \langle Ly, y \rangle.$$

2. If $\theta \neq 0$ then $D(Q) = H^1(\mathbb{R}_+) \oplus \mathfrak{h}$ and

$$Q : H^1(\mathbb{R}_+) \oplus \mathfrak{h} \to \mathbb{R}, \qquad Q(\phi, y) = \|\phi'\|_2^2 - \langle Ly, y \rangle - \frac{1}{\theta} |\phi(0_+) - \langle w, y \rangle|^2.$$

Proof. For any $(\phi, y) \in D(A)$ one has

$$\begin{aligned} Q(\phi, y) &= \|\phi'\|_2^2 - \langle Ly, y \rangle + (\phi')^*(0_+)(\phi(0) - \langle w, y \rangle) \\ &= \|\phi'\|_2^2 - \langle Ly, y \rangle - \theta |\phi'(0_+)|^2. \end{aligned}$$

Thus, the proof is completed if Q is bounded from below and closed. This follows from

$$|\phi(0_+)|^2 \leqslant a\|\phi\|_2^2 + b\|\phi'\|_2^2, \qquad a > 0, \quad 0 < b < 1.$$

□

Let us make $D(Q) \subset L^2(\mathbb{R}^3) \oplus \mathfrak{h}$ a Banach space with norm

$$\|(\phi, y)\|_Q^2 := Q(\phi, y) + (\sup \sigma(A) + 1)\big(\|\phi\|_2^2 + \|y\|^2\big)$$

and define

$$\mathcal{H}_\circ := D(Q) \oplus L^2(\mathbb{R}_+) \oplus \mathfrak{h}.$$

Then one has the following:

Theorem 3.2. *The linear operator,*

$$\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} : D(A) \oplus D(Q) \subset \mathcal{H}_\circ \to \mathcal{H}_\circ,$$

is the generator of a strongly continuous group of evolution

$$U_\circ^t : \mathcal{H}_\circ \to \mathcal{H}_\circ$$

which preserves the energy

$$E((\phi, y), (\dot\phi, \dot y)) := \tfrac{1}{2}\big(Q(\phi, y) + \|\dot\phi\|_2^2 + \|\dot y\|^2\big).$$

Such an operator is the Hamiltonian linear vector field corresponding to the quadratic Hamiltonian E via the canonical symplectic form on $L^2(\mathbb{R}_+) \oplus \mathfrak{h} \oplus L^2(\mathbb{R}_+) \oplus \mathfrak{h}$ given by

$$\Omega((\phi_1, y_1, \dot\phi_1, \dot y_1), (\phi_2, y_2, \dot\phi_2, \dot y_2)) := \langle \phi_1, \dot\phi_2 \rangle_2 - \langle \phi_2, \dot\phi_2 \rangle_2 + \langle y_1, \dot y_2 \rangle - \langle y_2, \dot y_2 \rangle.$$

Proof. The operator A is self-adjoint and bounded from above. Thus, the result concerning evolution generation follows from the theory of abstract wave equations (see e.g. [6], chapter 2, section 7). The results about the Hamiltonian structure follow from the theory of linear Hamiltonian systems in infinite dimensions (see e.g. [5], chapter 2). □

Remark 3.3. The above results can be immediately extended to a nonlinear situation. Indeed, given the potential function V let us consider the Hamiltonian

$$H : \mathcal{H}_\circ \to \mathbb{R},$$

where

$$H((\phi, y), (\dot\phi, \dot y)) = \tfrac{1}{2}\left(\|\phi'\|_2^2 + \|\dot\phi\|_2^2 + \|\dot y\|^2\right) + V(y)$$

when $\theta = 0$ and

$$H((\phi, y), (\dot\phi, \dot y)) = \frac{1}{2}\left(\|\phi'\|_2^2 + \|\dot\phi\|_2^2 + \|\dot y\|^2 - \frac{1}{\theta}|\phi(0_+) - \langle w, y\rangle|^2\right) + V(y)$$

when $\theta \neq 0$.

The nonlinear Hamiltonian vector field corresponding, via the canonical symplectic form on $L^2(\mathbb{R}_+) \oplus \mathfrak{h} \oplus L^2(\mathbb{R}_+) \oplus \mathfrak{h}$, to H is given by

$$X_H : D(A) \oplus D(Q) \subset \mathcal{H}_\circ \to \mathcal{H}_\circ,$$
$$X_H((\phi, y), (\dot\phi, \dot y)) := (\dot\phi, \dot y, \phi'', -\nabla V(y) + w\phi'(0_+)).$$

Obviously $X_H = X_E + B$, where X_E is the linear Hamiltonian vector field corresponding to the quadratic Hamiltonian E and B is vector field $B((\phi, y), (\dot\phi, \dot y)) := (0, 0, -(Ly + \nabla V(y)), 0)$. Thus, if V is twice differentiable then by Segal's existence theorem (see [17]) X_H generates a local continuous nonlinear symplectic flow on $\mathcal{H}_\circ$. Since Q is bounded from below, if

$$V(y) \geqslant c_1\|y\|^2 - c_2, \qquad c_1 > 0, \quad c_2 \geqslant 0,$$

then such a flow is global.

4. Examples

Example 4.1. The dynamics of the Lamb model (see [10]), given by the equations

$$\ddot\phi(t, x) = \phi''(t, x)$$
$$M\ddot y(t) = -Ky(t) + T\phi'(t, 0_+)$$
$$y(t) = \phi(t, 0_+),$$

is described by the self-adjoint extension A corresponding to

$$\dim \mathfrak{h} = 1, \qquad \langle x, y\rangle = \frac{M}{T}x^* y, \qquad Ly = -\frac{K}{M}y, \qquad w = \frac{T}{M}, \qquad \theta = 0.$$

The similar model with n point masses

$$\ddot\phi(t, x) = \phi''(t, x)$$
$$M_1\ddot y_1(t) = -K_1(y_1(t) - y_2(t)) + T\phi'(t, 0_+)$$
$$M_2\ddot y(t)_2 = -K_2(y_2(t) - y_3(t)) + K_1(y_1(t) - y_2(t))$$
$$\vdots$$
$$M_{n-1}\ddot y_{n-1}(t) = -K_{n-1}(y_{n-1}(t) - y_n(t)) + K_{n-2}(y_{n-2}(t) - y_{n-1}(t))$$
$$M_n\ddot y_n(t) = -K_n y_n(t) + K_{n-1}(y_{n-1}(t) - y_n(t))$$
$$y_1(t) = \phi(t, 0_+)$$

is described by the self-adjoint extension A corresponding to

$$\dim \mathfrak{h} = n, \qquad \langle x, y\rangle = \frac{1}{T}\sum_{j=1}^{n} M_j x_j^* y_j,$$

$$L = \begin{pmatrix} -\frac{K_1}{M_1} & \frac{K_1}{M_1} & 0 & 0 & \cdots & 0 \\ \frac{K_1}{M_2} & -\frac{K_1+K_2}{M_2} & \frac{K_2}{M_2} & 0 & \cdots & 0 \\ 0 & \frac{K_2}{M_3} & -\frac{K_2+K_3}{M_3} & \frac{K_3}{M_3} & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & \frac{K_{n-3}}{M_{n-2}} & -\frac{K_{n-3}+K_{n-2}}{M_{n-2}} & \frac{K_{n-2}}{M_{n-2}} & 0 \\ 0 & \cdots & 0 & \frac{K_{n-2}}{M_{n-1}} & -\frac{K_{n-2}+K_{n-1}}{M_{n-1}} & \frac{K_{n-1}}{M_{n-1}} \\ 0 & \cdots & 0 & 0 & \frac{K_{n-1}}{M_n} & -\frac{K_{n-1}+K_n}{M_n} \end{pmatrix},$$

$$w = \left(\frac{T}{M_1}, 0, \ldots, 0\right), \qquad \theta = 0.$$

The corresponding Hamiltonian is given by

$$H : \{(\phi, y) \in H^1(\mathbb{R}_+) \oplus \mathbb{C}^n : \phi(0_+) = y_1\} \oplus L^2(\mathbb{R}_+) \oplus \mathbb{C}^n \to \mathbb{R}$$

$$H((\phi, y), (\dot\phi, \dot y)) := \frac{1}{2}\left(\|\dot\phi\|_2^2 + \|\phi'\|_2^2 + \frac{1}{T}\sum_{k=1}^{n} M_k|\dot y_k|^2 + \frac{1}{T}\Lambda y\cdot y\right),$$

where the matrix Λ is given by

$$\Lambda = \begin{pmatrix} K_1 & -K_1 & 0 & \cdots & 0 \\ -K_1 & K_1+K_2 & -K_2 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & -K_{n-2} & K_{n-2}+K_{n-1} & -K_{n-1} \\ 0 & \cdots & 0 & -K_{n-1} & K_{n-1}+K_n \end{pmatrix}$$

and $\cdot$ denotes the standard inner product in $\mathbb{C}^n$.

In the following examples we describe models, from classical electrodynamics and theoretical acoustic respectively, which are not interpretable as standard Lamb models in that $\theta \neq 0$.

Example 4.2. The renormalized Pauli–Fierz model.

A three-dimensional charged oscillator characterized by frequency ω, mass m and electric charge e interacting with the electromagnetic field in dipole approximation has a dynamics described, in the point limit and after mass renormalization, by a well-defined self-adjoint operator which couples particle momentum and vector electromagnetic potential. Dynamics and its main properties, classical and quantum, are constructed and studied in [3]. This is the point limit of the Pauli–Fierz model for the case of a quadratic potential energy. Due to the dipole approximation, the action of this operator is nontrivial (i.e. different from the free uncoupled dynamics) only on the radial component of the field, and by standard decomposition using vector spherical harmonics it turns out that on this monopole subspace the restricted dynamics for every couple (ϕ, p) constituted by a component of the vector potential on the nontrivial subspace, and a corresponding component of the particle momentum, is given by the coupled system

$$\ddot{\phi}(t,r) = \phi''(t,r)$$
$$\ddot{p}(t) = -\frac{3m}{2e}\omega^2\phi(t,0_+)$$
$$\phi'(t,0_+) + \frac{3m}{2e^2}\phi(t,0_+) = \frac{1}{e}p(t).$$

Writing the field ϕ which appears in the evolution equation for p in terms of its derivative ϕ' and p by means of the boundary condition, one obtains

$$\ddot{\phi}(t,r) = \phi''(t,r)$$
$$\ddot{p}(t) = -\omega^2 p(t) + e\omega^2\phi'(t,0_+)$$
$$\frac{2e^2}{3m}\phi'(t,0_+) + \phi(t,0_+) = \frac{2e}{3m}p(t).$$

Let us remark here that by Newton's law $\dot{p} = -m\omega^2 q$, where q is a component of the particle position. Thus, the Cauchy initial datum for $\dot{p}$ is obtained from the initial position.

In conclusion, the dynamics of the renormalized Pauli–Fierz model in dipole approximation and with quadratic external potential is described by the self-adjoint operator A corresponding to

$$\dim\mathfrak{h} = 1, \qquad \langle x, y\rangle = \frac{2x^*y}{3m\omega^2}, \qquad Ly = -\omega^2 y, \qquad w = e\omega^2, \qquad \theta = \frac{2e^2}{3m}.$$

The corresponding Hamiltonian is given by

$$H : H^1(\mathbb{R}_+)\oplus\mathbb{C}\oplus L^2(\mathbb{R}_+)\oplus\mathbb{C}\to\mathbb{R}$$
$$H((\phi,p),(\dot{\phi},\dot{p})) := \frac{1}{2}\left(\|\dot{\phi}\|_2^2 + \|\phi'\|_2^2\right) + \frac{1}{3m}\left(\frac{|\dot{p}|^2}{\omega^2} + |p|^2 - \left|\frac{3m}{2e}\phi(0_+) - p\right|^2\right).$$

As recalled in the introduction, a field–particle interaction which reduces to the standard ($\theta = 0$) Lamb model in the point limit and after *spring constant* renormalization is the Schwabl–Thirring model, in which a scalar field interacts with a scalar oscillator (for details, in a different framework, see [12]).

Example 4.3. A spherical elastic shell in the acoustic field.

Let us consider the exterior problem for a spherical shell of mass M, radius R_0 and constant surface density $\rho = \frac{M}{4\pi R_0^2}$ undergoing radial motion only and interacting with an irrotational acoustic field in the linear approximation. The shell is elastic, i.e. on every surface element acts a restoring force proportional to the radius, and of Young modulus K. If small radial oscillations around R_0 are considered, introducing the variables ψ, related to the acoustic potential ϕ by $\phi(R_0 + r) = \frac{\psi(r)}{r}$, $r > 0$, and the radius $R_0 + R(t)$ and taking into account the continuity of velocity ϕ' of the acoustic field at the boundary, one obtains the equations of motion (assuming propagation velocity equal to one)

$$\ddot{\psi}(t,r) = \psi''(t,r)$$
$$M\ddot{R}(t) = -KR(t) + 4\pi R_0\rho\dot{\psi}(t,0_+)$$
$$\frac{\psi'(t,0_+)}{R_0} - \frac{\psi(t,0_+)}{R_0^2} = \dot{R}(t).$$

This system can be converted to a generalized Lamb system by introducing the new variable (in fact a sort of total momentum)

$$P := M\dot{R} - 4\pi R_0\rho\psi(0_+).$$

Rewriting the above equations in terms of the new variable P, expressing the field ψ which appears in the evolution equation for P in terms of its derivative ψ' and P by means of the boundary condition, one obtains

$$\ddot{\psi}(t,r) = \psi''(t,r)$$

$$\ddot{P}(t) = -\frac{K}{M+4\pi R_0^3\rho}P(t) - \frac{4\pi K R_0^2\rho}{M+4\pi R_0^3\rho}\psi'(t,0_+) - \frac{MR_0}{M+4\pi R_0^3\rho}\psi'(t,0_+) + \psi(t,0_+)$$
$$= -\frac{R_0^2}{M+4\pi R_0^3\rho}P(t).$$

As regards the initial datum for $\dot{P}$ it can be recovered from the one for R as in the previous example.

By defining $\omega^2 := \frac{K}{M}$, the above system is described by the self-adjoint operator A corresponding to

$$\dim \mathfrak{h} = 1, \qquad \langle x, y\rangle = \frac{x^* y}{4\pi K}, \qquad Ly = -\frac{\omega^2}{1+R_0}y,$$
$$w = -\frac{4\pi\omega^2 R_0^2}{1+R_0}, \qquad \theta = -\frac{R_0}{1+R_0}.$$

The corresponding Hamiltonian is given by

$$H : H^1(\mathbb{R}_+) \oplus \mathbb{C} \oplus L^2(\mathbb{R}_+) \oplus \mathbb{C} \to \mathbb{R}$$

$$H((\psi, P), (\dot{\psi}, \dot{P})) := \frac{1}{2}\left(\|\dot{\psi}\|_2^2 + \|\psi'\|_2^2\right) + \frac{1}{8\pi M}\left(\frac{|\dot{P}|^2}{\omega^2} + \frac{|P|^2}{1+R_0}\right)$$
$$+ \frac{1+R_0}{2R_0}\left|\psi(0_+) + \frac{R_0^2}{1+R_0}\frac{P}{M}\right|^2.$$

An analysis of interaction of elastic surfaces with acoustic fields from the point of view of Lax–Phillips scattering theory is given in [2]. For the study of one-dimensional models in acoustics in the framework of self-adjoint extensions we refer to [4].

5. Wave equations with high-order boundary conditions

From now on we will consider self-adjoint operators A which are self-adjoint extensions corresponding (according to theorem 2.1) to L's and w's such that

$$\{L^k w\}_0^{n-1} \quad \text{is a basis in } \mathfrak{h}. \tag{5.1}$$

Note that the examples given in section 4 satisfy such hypothesis.

With respect to the orthonormal base obtained from $\{L^k w\}_0^{n-1}$ by the Schmidt orthogonalization procedure, the linear operator L is represented by a Jacobi matrix. However, we prefer to consider here the unitary isomorphism $\mathfrak{h} \simeq \mathbb{C}^n$ induced by the orthonormal system $\{\hat{e}_i\}_1^n$ made of the eigenvectors of L. For any vector $y \in \mathfrak{h}$ and for any linear operator $M : \mathfrak{h} \to \mathfrak{h}$ we pose

$$y \equiv (y_1, \ldots, y_n), \qquad M \equiv \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \cdots & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix}.$$

With these notation

$$w \equiv (w_1, \dots, w_n), \qquad L \equiv \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix},$$

where $\sigma(L) = \{\lambda_1, \dots, \lambda_n\}$. We introduce the diagonal matrix

$$W \equiv \begin{pmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & w_n \end{pmatrix},$$

the Vandermonde matrix

$$V \equiv \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix}$$

and then we define

$$M := VW.$$

Since

$$\det M = \prod_{1 \leqslant i < j \leqslant n} (\lambda_j - \lambda_i) \prod_{1 \leqslant i \leqslant n} w_i \neq 0,$$

our hypothesis (5.1) is equivalent to

$$\lambda_i \neq \lambda_j \qquad \text{and} \qquad \sigma_w(L) = \emptyset.$$

Thus under our hypothesis the spectrum of L is simple and, by theorem 2.2, A has no eigenvalue immersed in the continuous spectrum.

Let us denote by $\mathcal{S}(\mathbb{R}_+)$ the space of rapidly decreasing smooth functions on $[0, +\infty)$. We define the dense subspace $\mathcal{D} \subset \mathcal{H}_\circ$ by

$$\mathcal{D} := \{(\phi, y, \dot\phi, \dot y) \in \mathcal{H}_\circ : \phi \in \mathcal{S}(\mathbb{R}_+), \dot\phi \in \mathcal{S}(\mathbb{R}_+), \\ y = M^{-1} v(\phi), \dot y = M^{-1} v(\dot\phi)\},$$

where

$$v(\phi) = \sum_{k=1}^{n} p_k(\partial_x) \phi(0_+) \hat e_k,$$

and $p_k(\partial_x)$ is the differential operator with constant coefficients associated with the polynomial recursively defined by

$$p_1(z) = \theta z + 1, \qquad p_k(z) = z^2 p_{k-1}(z) - \langle w, L^{k-2} w \rangle z, \quad k \geqslant 2.$$

The next theorem is the main technical point as regards the successive study of the Lax–Phillips scattering of generalized Lamb models. It says that $\mathcal{D}$ is invariant under the flow $U_\circ^t$ and that on this dense subspace a generalized Lamb model is equivalent to a wave equation with a high-order boundary condition at zero.

Theorem 5.1. *Let $U_\circ^t$ be the strongly continuous group of evolution provided by theorem 3.2. Then,*

$$U_\circ^t : \mathcal{D} \to \mathcal{D}$$

and

$$U_\circ^t(\phi_0, M^{-1}v(\phi_0), \dot\phi, M^{-1}v(\dot\phi_0)) = (\phi(t), M^{-1}v(\phi(t)), \dot\phi(t), M^{-1}v(\dot\phi(t)))$$

where $\phi(t,x)$ solves the equations

$$\begin{aligned}
&\partial_{tt}\phi(t,x) = \partial_{xx}\phi(t,x), \qquad x > 0,\\
&p(\partial_x)\phi\,(t, 0_+) = 0\\
&\phi(0,x) = \phi_0(x), \qquad \dot\phi(0,x) = \dot\phi_0(x).
\end{aligned} \tag{5.2}$$

Here, $p(\partial_x)$ denotes the constant coefficients differential operator of degree $2n+1$ ($2n$ if $\theta = 0$) associated with the polynomial

$$p(z) = p_{n+1}(z) - \sum_{i,j=1}^{n} \lambda_i^n (V^{-1})_{ij} p_j(z).$$

Proof. Let $(\phi(t), \dot\phi(t))$ be the solution of the Cauchy problem

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\phi(t), y(t)) = A(\phi(t), y(t)),\\
&(\phi(0), y(0)) = (\phi_0, y_0),\\
&(\dot\phi(0), \dot y(0)) = (\dot\phi_0, \dot y_0),
\end{aligned} \tag{5.3}$$

with $(\phi_0, y_0, \dot\phi_0, \dot y_0) \in \mathcal{D}$ and let us suppose that $(\phi(t), \dot\phi(t))$ is in $\mathcal{S}(\mathbb{R}_+)$ for all times. Then deriving with respect to time the boundary condition $2k$-times, $k = 0, \ldots, n$, using $\ddot\phi(t, 0_+) = \phi''(t, 0_+)$, one obtains the $n+1$ equations

$$\langle w, L^k y(t)\rangle = p_{k+1}(\partial_x)\phi(t, 0_+), \qquad k = 0, 1, \ldots, n.$$

The first n of such equations can be rewritten as

$$My(t) = v(\phi(t))$$

so that

$$y(t) = M^{-1}v(\phi(t)), \qquad \dot y(t) = M^{-1}v(\dot\phi(t))$$

for all times. Moreover, inserting the expression for $y(t)$ into the nth equation one obtains $p(\partial_x)\phi(t, 0_+) = 0$, so ϕ satisfies (5.2).

Conversely, let $\phi(t)$ be the solution of (5.2) and put

$$y(t) := M^{-1}v(\phi(t)).$$

Then, the n equations

$$\langle w, L^k y(t)\rangle = p_{k+1}(\partial_x)\phi(t, 0_+), \qquad k = 0, 1, \ldots, n-1.$$

are satisfied. The first equation says that (ϕ, y) and $(\dot\phi, \dot y)$ are in $D(A)$. Deriving each equation two times with respect to time and using $\ddot\phi(t, 0_+) = \phi''(t, 0_+)$, one obtains

$$\begin{aligned}
\langle w, L^k \ddot y(t)\rangle &= p_{k+1}(\partial_x)\phi''(t, 0_+) = p_{k+2}(\partial_x)\phi(t, 0_+) + \langle w, L^k w\rangle\phi'(t, 0_+)\\
&= \langle w, L^k(Ly(t) + w\phi'(t, 0_+))\rangle, \qquad k = 0, 1, \ldots, n-1,
\end{aligned}$$

which implies $\ddot{y}(t) = Ly(t) + w\phi'(t, 0_+)$. So (ϕ, y) is the solution of (5.3). This also shows, by unicity, that if the initial conditions of (5.3) are in $\mathcal{S}(\mathbb{R}_+)$ then they are in $\mathcal{S}(\mathbb{R}_+)$ for all times. This justifies the assumption we made at the beginning of the proof. □

The next lemma makes the polynomial p appearing in the previous theorem more explicit.

Lemma 5.2.

$$p(z) = (\theta z + 1)\det(z^2 - L) - z\sum_{j=1}^{n}\left(\sum_{k=1}^{j} a_{j-k}\langle w, L^{k-1}w\rangle\right) z^{2(n-j)},$$

where

$$a_0 = 1, \qquad a_j := (-1)^j \sum_{i_1<\cdots<i_j} \lambda_{i_1}\cdots\lambda_{i_j}, \qquad 1 \leqslant j \leqslant n.$$

Proof. Put

$$\tilde{a}_j := -\sum_{i=1}^{n} \lambda_i^n (V^{-1})_{ij}, \qquad 1 \leqslant j \leqslant n, \qquad \tilde{a}_{n+1} := 1$$

and

$$b_{jk} := -\sum_{i=1}^{n} \lambda_i^{k-1-j}|w_i|^2, \qquad 1 \leqslant k \leqslant n+1, \quad 1 \leqslant j \leqslant k-1.$$

By the definitions of $p(z)$, $p_k(z)$, $\tilde{a}_j$ and b_{jk}, one has

$$\begin{aligned} p(z) &= (z\theta + 1)\sum_{j=1}^{n+1}\tilde{a}_j z^{2(j-1)} + z\left(\sum_{j=1}^{n} b_{j,n+1}z^{2(j-1)} + \sum_{j=2}^{n}\tilde{a}_j\sum_{i=1}^{j-1} b_{ij}z^{2(j-1)}\right) \\ &= (z\theta + 1)\sum_{j=1}^{n+1}\tilde{a}_j z^{2(j-1)} + z\sum_{j=1}^{n}\left(\sum_{k=j+1}^{n+1} b_{jk}\tilde{a}_k\right) z^{2(j-1)} \\ &= (z\theta + 1)\, p_a(z^2) + z p_b(z^2), \end{aligned}$$

where

$$p_a(z) = \sum_{j=1}^{n+1}\tilde{a}_j z^{j-1} \equiv \sum_{j=0}^{n} a_j z^{n-j},$$

$$p_b(z) = \sum_{j=1}^{n}\tilde{b}_j z^{j-1} \equiv \sum_{j=1}^{n} b_j z^{n-j},$$

$$\tilde{b}_j = -\sum_{k=1}^{n}\sum_{i=j+1}^{n+1}\lambda_k^{i-j-1}\tilde{a}_i|w_k|^2 = -\sum_{i=j+1}^{n+1}\tilde{a}_i\langle w, L^{i-j-1}w\rangle.$$

Since

$$\sum_{j=1}^{n+1}\tilde{a}_j\lambda_k^{j-1} = \lambda_k^n - \sum_{j=1}^{n}\sum_{i=1}^{n}\lambda_i^n (V^{-1})_{ij}V_{jk} = \lambda_k^n - \lambda_k^n = 0,$$

the eigenvalues $\lambda_1, \ldots, \lambda_n$ are the roots of p_a. Thus,

$$\tilde{a}_j = \sum_{i_1<\cdots<i_{n-j+1}} (-1)^{n-j+1} \lambda_{i_1} \cdots \lambda_{i_{n-j+1}}, \qquad 1 \leqslant j \leqslant n, \quad \tilde{a}_{n+1} = 1.$$

□

The next result gives the relation between the roots of p and eigenvalues and resonances of the self-adjoint extension A.

Lemma 5.3. *Suppose* $\det L \neq 0$ *and let us define the couple* (ϕ, y) *by*

$$\phi(x) := \mathrm{e}^{zx}, \qquad y := M^{-1} v(\phi).$$

Then,

$$p(z) = 0 \quad \Longleftrightarrow \quad \begin{cases} \phi'' = z^2 \phi, \\ Ly + w\phi'(0_+) = z^2 y \\ \theta\phi'(0_+) + \phi(0_+) = \langle w, y \rangle, \end{cases} \quad \Longrightarrow \quad z \in \mathbb{C} \backslash \mathrm{i}\mathbb{R}.$$

Hence,

$$p(z) = 0, \qquad \mathrm{Re}(z) \leqslant 0, \quad \Longleftrightarrow \quad z = -\sqrt{\lambda}, \qquad \lambda \in \sigma_{\mathrm{pp}}(A) \cap (0, +\infty).$$

Proof. Since $\phi'' = z^2\phi$ implies $z \neq 0$ and $p(0) = (-1)^n \det L \neq 0$, we can take $z \neq 0$. By the definition of (ϕ, y), we only need to show that

$$p(z) = 0 \quad \Longleftrightarrow \quad Ly + wz = z^2 y^2.$$

Moreover, beside $\langle w, y \rangle = p_1(z)$, one has

$$\langle w, L^k y \rangle = p_{k+1}(z) = z^2 p_k(z) - \langle w, L^{k-1} w \rangle z, \qquad k = 1, \ldots, n-1.$$

The above equalities, together with $p(z) = 0$, give

$$\langle w, L^n y \rangle = p_{n+1}(z) = z^2 p_n(z) - \langle w, L^{n-1} w \rangle z$$

Thus,

$$\langle w, L^k (Ly + wz - z^2 y) \rangle = 0, \qquad k = 0, \ldots, n-1,$$

i.e., $Ly + wz = z^2 y$. Reversing the above argument one has that $Ly + wz = z^2 y$ implies $p(z) = 0$.

Suppose now $p(\mathrm{i}\nu) = 0$, $\nu \in \mathbb{R}$, so that $Ly + \mathrm{i}\nu w = -\nu^2 y$. Since $\langle w, y \rangle = \mathrm{i}\nu\theta + 1$ we have

$$\langle (L + \nu^2) y, y \rangle = \nu^2 \theta - \mathrm{i}\nu.$$

Since $L + \nu^2$ is symmetric we have $\nu = 0$. But $\nu \neq 0$ by $\det L \neq 0$. □

Remark 5.4. By the previous lemma we have that the polynomial p has no purely imaginary roots. Those in the left half plane are real and correspond to eigenvalues, and those in the right half plane give rise to non-normalizable solutions of the eigenvalue equations and correspond to resonant states.

6. Lax–Phillips scattering

From now on we suppose that

$$\sigma_{pp}(A) = \emptyset. \tag{6.1}$$

Here we remark that the successive results hold, with the appropriate modifications, even without this hypothesis which is just a convenient choice in order to simplify the exposition.

Since we already supposed $\sigma_w(L) = \emptyset$, by theorem 2.2, the above hypothesis means

$$\{\lambda \in \rho(L) \cap (0, +\infty) : \theta + \Gamma(\lambda) = 0\} = \emptyset,$$

i.e., we are supposing that there are no strictly positive solutions x of the equation

$$\frac{1}{x} + \sum_{j=1}^{n} \frac{|w_j|^2}{-\lambda_j + x^2} = \theta.$$

This is true if and only if

$$\sigma(L) \subset (-\infty, 0), \qquad \theta \leqslant 0.$$

Thus, hypothesis (6.1) is satisfied by both examples 4.1 and 4.3.

By lemma 5.3, (6.1) implies that the polynomial p has no negative real root and that the complex ones (which appear in complex-conjugate pairs since p has real coefficients) are all contained in the right half plane. Thus,

$$\sigma_{pp}(A) = \emptyset \quad \Longleftrightarrow \quad \mathrm{Roots}(p) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}.$$

By functional calculus, since A is injective and negative by (6.1) we have that

$$U_\circ^t = \begin{pmatrix} \cos t\sqrt{-A} & \sqrt{-A}^{-1} \sin t\sqrt{-A} \\ -\sqrt{-A} \sin t\sqrt{-A} & \cos t\sqrt{-A} \end{pmatrix}.$$

Moreover $U_\circ^t$ extends to a strongly continuous unitary group

$$U^t : \mathcal{H} \to \mathcal{H},$$

where $\mathcal{H}$ is the Hilbert space given by the completion of $\mathcal{H}_\circ$ with respect to the scalar product corresponding to energy norm

$$\|(\phi, y, \dot{\phi}, \dot{y})\|_E := E(\phi, y, \dot{\phi}, \dot{y})^{1/2},$$

Remark 6.1. By theorem 2.2 our hypothesis $\sigma_{pp}(A) = \emptyset$ says that A has purely absolutely continuous spectrum. Thus,

$$\lim_{t\to\pm\infty} \|y(t)\| = 0.$$

Indeed by functional calculus and Riemann–Lebesgue lemma, denoting by $P(d\lambda)$ the projection-valued measure corresponding to A, one has, for any $(\phi, y, \dot{\phi}, \dot{y}) \in \mathcal{H}_\circ$,

$$\lim_{t\to\pm\infty} \lambda_i y_i(t) = \lim_{t\to\pm\infty} \langle\sqrt{-A}(0, \hat{e}_i), \sqrt{-A} \cos t\sqrt{-A}(\phi, y) + \sin t\sqrt{-A}(\dot{\phi}, \dot{y})\rangle_{L^2(\mathbb{R}_+)\oplus\mathfrak{h}}$$
$$= \lim_{t\to\pm\infty} \int_0^\infty \cos t\sqrt{\lambda}\langle\sqrt{-A}(0, \hat{e}_i), P(d\lambda)\sqrt{-A}(\phi, y)\rangle_{L^2(\mathbb{R}_+)\oplus\mathfrak{h}}$$
$$+ \lim_{t\to\pm\infty} \int_0^\infty \sin t\sqrt{\lambda}\langle\sqrt{-A}(0, \hat{e}_i), P(d\lambda)(\dot{\phi}, \dot{y})\rangle_{L^2(\mathbb{R}_+)\oplus\mathfrak{h}} = 0$$

Analogously, one has

$$\lim_{t\to\pm\infty} \|\dot{y}(t)\| = 0.$$

Define

$$\mathcal{F} := \{(f_-, f_+) \in \mathcal{S}_0(\mathbb{R}) \times \mathcal{S}_0(\mathbb{R}) : p(\partial_x) f_- + p(-\partial_x) f_+ = 0\},$$

where

$$\mathcal{S}_0(\mathbb{R}) := \left\{ f \in \mathcal{S}(\mathbb{R}) : \int_{\mathbb{R}} f(x)\,\mathrm{d}x = 0 \right\}$$
$$\equiv \{ f \in \mathcal{S}(\mathbb{R}) : f = g',\, g \in \mathcal{S}(\mathbb{R})\}.$$

By considering the Fourier transform (denoted by $\hat{}$) of the differential equation

$$p(\partial_x) f_- + p(-\partial_x) f_+ = 0, \tag{6.2}$$

one obtains

$$p(\mathrm{i}\kappa)\hat{f}_-(\kappa) + p(-\mathrm{i}\kappa)\hat{f}_+(\kappa) = 0.$$

Thus one has the following result, which permits to define what will be the scattering operator.

Lemma 6.2.

$$(f_-, f_+) \in \mathcal{F} \qquad \Longleftrightarrow \qquad \hat{f}_+(\kappa) = -\frac{p(\mathrm{i}\kappa)}{p(-\mathrm{i}\kappa)}\hat{f}_-(\kappa).$$

Since p has real coefficients, we have

$$\left| \frac{p(\mathrm{i}\kappa)}{p(-\mathrm{i}\kappa)} \right| = 1,$$

therefore

$$S_p : \mathcal{S}_0(\mathbb{R}) \to \mathcal{S}_0(\mathbb{R}), \qquad (S_p f)\hat{}\,(k) := -\frac{p(\mathrm{i}\kappa)}{p(-\mathrm{i}\kappa)}\hat{f}(\kappa)$$

extends to a unitary map on $L^2(\mathbb{R})$.

Let $\phi(t)$ be the solution of (5.2) with initial data $(\phi, y, \dot{\phi}, \dot{y}) \in \mathcal{D}$. Then,

$$\phi(t, x) = a(x + t) + b(t - x),$$

where the couple (a, b) is determined on a half line up to a constant c by

$$a(x) = -\frac{1}{2}\int_x^{+\infty} (\dot{\phi}(y) + \phi'(y))\,\mathrm{d}y + c, \qquad x \geqslant 0, \tag{6.3}$$

$$b(-x) = \frac{1}{2}\int_x^{+\infty} (\dot{\phi}(y) - \phi'(y))\,\mathrm{d}y - c, \qquad x \geqslant 0. \tag{6.4}$$

The functions $a(x)$ and $b(-x)$ are then determined for the remaining values of $x < 0$ by solving the differential equation

$$p(\partial_x)a + p(-\partial_x)b = 0. \tag{6.5}$$

The following central result holds.

Theorem 6.3. *The map* $I_\circ \equiv (I_\circ^-, I_\circ^+)$ *defined by*

$$I_\circ : \mathcal{D} \to \mathcal{F}, \qquad I_\circ(\phi, y, \dot{\phi}, \dot{y}) := (f_-, f_+), \qquad f_- = a', \qquad f_+ = b'.$$

is one to one and extends to a unitary map

$$I : \mathcal{H} \to \mathrm{Graph}(S_p).$$

Proof. It is easy to check that the map $I_\circ : \mathcal{D} \to \mathcal{F}$ is injective. It is surjective too. Indeed, if $(f_+, f_-) \in \mathcal{F}$ then $I_\circ(\phi, y, \dot\phi, \dot y) = (f_+, f_-)$ where

$$(\phi, y, \dot\phi, \dot y) = (\phi, M^{-1}v(\phi), \dot\phi, M^{-1}v(\dot\phi)),$$
$$\phi(x) = a(x) + b(-x), \qquad \dot\phi(x) = a'(x) + b'(-x),$$
$$a(x) := \int_{-\infty}^{x} f_-(y)\,\mathrm{d}y, \qquad b(x) := \int_{-\infty}^{x} f_+(y)\,\mathrm{d}y.$$

Let us now show that

$$\|(\phi, y, \dot\phi, \dot y)\|_E^2 = \|I_\circ^-(\phi, y, \dot\phi, \dot y)\|_2^2 + \|I_\circ^+(\phi, y, \dot\phi, \dot y)\|_2^2.$$

Since

$$I_\circ^+ = S_p I_\circ^-$$

and S_p is unitary we need to show that

$$\|(\phi, y, \dot\phi, \dot y)\|_E^2 = 2\|I_\circ^-(\phi, y, \dot\phi, \dot y)\|_2^2.$$

Let $(\phi(t,x), y(t), \dot\phi(t,x), \dot y(t))$ the solution of

$$\begin{cases} \ddot\phi(t,x) - \phi''(t,x) = 0 \\ \ddot y(t) - Ly(t) - w\phi'(t,0_+) = 0 \\ \theta\phi'(t,0_+) + \phi(t,0_+) - \langle w, y(t)\rangle = 0 \end{cases} \tag{6.6}$$

with initial data $(\phi(x), y, \dot\phi(x), \dot y)$. Then, by using the evolution equation for y and the boundary conditions, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\dot y(t)\|^2 - \langle y(t), Ly(t)\rangle - \theta|\phi'(t,0_+)|^2) = \dot\phi^*(t,0_+)\phi'(t,0_+) + (\phi')^*(t,0_+)\dot\phi(t,0_+) \tag{6.7}$$

and so

$$\int_0^\infty (\dot\phi^*(t,0_+)\phi'(t,0_+) + (\phi')^*(t,0_+)\dot\phi(t,0_+))\,\mathrm{d}t = -(\|\dot y\|^2 - \langle y, Ly\rangle - \theta|\phi'(0_+)|^2).$$

By conservation of energy, one has

$$\begin{aligned} 2E(\phi, y, \dot\phi, \dot y) &= \|\dot\phi\|_2^2 + \|\phi'\|_2^2 + \|\dot y\|^2 - \langle y, Ly\rangle - \theta|\phi'(0_+)|^2 \\ &= \int_0^\infty (\dot\phi^*(x)\dot\phi(x) + (\phi')^*(x)\phi'(x))\,\mathrm{d}x \\ &\quad - \int_0^\infty (\dot\phi^*(t,0_+)\phi'(t,0_+) + (\phi')^*(t,0_+)\dot\phi(t,0_+))\,\mathrm{d}t. \end{aligned}$$

Inserting into the last equation $\phi(x,t) = a(t+x) + b(t-x)$ and using $b' = S_p a'$, we have

$$E(\phi, y, \dot\phi, \dot y) = 2\|b'\|_2^2 = 2\|I_\circ^-(\phi, y, \dot\phi, \dot y)\|_2^2.$$

□

We now define the maps $R_\circ^\pm : \mathcal{D} \to \mathcal{S}_0(\mathbb{R})$ by

$$R_\circ^\pm(\phi, y, \dot\phi, \dot y)(x) := I_\circ^\pm(\phi, y, \dot\phi, \dot y)(-x)$$

and the orthogonal spaces $\mathcal{H}^\pm$ as the closure, with respect to the energy norm, of

$$\mathcal{D}^\pm := \left\{(\phi, y, \dot\phi, \dot y) \in \mathcal{D} : R_\circ^\pm(\phi, y, \dot\phi, \dot y) \in \mathcal{S}_0^\pm(\mathbb{R})\right\},$$

where

$$\mathcal{S}_0^{\pm}(\mathbb{R}) := \{f \in \mathcal{S}_0(\mathbb{R}) : f(x) = 0, \pm x \leqslant 0\}.$$

The next theorem shows that the subspaces $\mathcal{H}^-$ and $\mathcal{H}^+$ are incoming and outgoing in the sense of Lax–Phillips scattering theory (see [11, 15], section XI.11). The proof is a straightforward consequence of the previous definitions.

Theorem 6.4. *The subspace $\mathcal{H}^-$ is incoming and the subspace $\mathcal{H}^+$ is outgoing, i.e.,*

$$U^s\mathcal{H}^- \subset U^t\mathcal{H}^-, s < t \leqslant 0, \qquad \bigcap_{t<0} U^t\mathcal{H}^- = \{0\}, \qquad \overline{\bigcup_{t\in\mathbb{R}} U^t\mathcal{H}^-} = \mathcal{H},$$

$$U^t\mathcal{H}^+ \subset U^s\mathcal{H}^+, t > s \geqslant 0, \qquad \bigcap_{t>0} U^t\mathcal{H}^+ = \{0\}, \qquad \overline{\bigcup_{t\in\mathbb{R}} U^t\mathcal{H}^+} = \mathcal{H}.$$

By the previous theorem and by [11], chapter II, sections 2 and 3, there follows

Theorem 6.5. *The unitary maps*

$$R^{\pm} : \mathcal{H} \to L^2(\mathbb{R}),$$

defined as the closures of the maps $R_\circ^{\pm}$, provide incoming and outgoing translation representations of U^t, i.e.,

$$R^{\pm}U^t(R^{\pm})^{-1} = T^t, \qquad R^{\pm}\mathcal{H}^{\pm} = L^2(\mathbb{R}_{\pm}), \qquad S_p^* = R^+(R^-)^{-1},$$

where

$$T^t : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad T^t f(x) := f(x-t).$$

Proof. The thesis follows from theorem 6.3 and from simple computations. Otherwise one can use, as we said, the general theory contained in [11]. □

7. The Lax–Phillips semigroup

We are now interested in the evolution of the states which are neither incoming in the past nor outgoing in the future. To this end, one defines

$$Z^t := PU^tP,$$

where P is the orthogonal projection onto

$$\mathcal{K} := \mathcal{H} \ominus (\mathcal{H}^- \oplus \mathcal{H}^+).$$

Since $\mathcal{H}^+$ and $\mathcal{H}^-$ are orthogonal it is known (see [11, 15] section XI.11) that Z^t is a strongly continuous semigroup of contractions on $\mathcal{K}$ for positive times:

$$\forall t \geqslant 0, \qquad Z^t : \mathcal{K} \to \mathcal{K}, \qquad \|Z^t\| \leqslant 1, \qquad \lim_{t\uparrow\infty} Z^t = 0.$$

The next theorem completely characterizes such a semigroup (let us remark that, by (6.1), all the roots of p have positive real part).

Theorem 7.1. *The vector space $\mathcal{K}$ is finite dimensional,*

$$\dim \mathcal{K} = \deg(p).$$

It is generated by the vectors

$$(R^{+})^{-1}\phi_{kj}, \qquad k = 0, 1, \ldots, \nu_j - 1, \qquad j = 1, \ldots, m,$$
$$\phi_{kj}(x) := \begin{cases} x^k\, \mathrm{e}^{z_j x} & \text{for} \quad x \leqslant 0 \\ 0 & \text{for} \quad x > 0, \end{cases} \tag{7.1}$$

where $z_1, \ldots, z_m$ *are the roots of the polynomial* p *and* $\nu_1, \ldots, \nu_m$ *the respective multiplicities. Moreover,*

$$Z^t = \mathrm{e}^{-tB}, \qquad \sigma(B) = \{z_1, \ldots, z_m\},$$

and the matrix representing B *with respect to the basis (7.1) is the direct sum* $B = \oplus_{j=1}^{m} B_j$ *where* B_j *is the* $\nu_j \times \nu_j$ *matrix:*

$$\begin{pmatrix} z_j & 1 & 0 & \cdots & 0 \\ 0 & z_j & 2 & \cdots & \vdots \\ 0 & 0 & z_j & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \nu_j - 1 \\ 0 & 0 & \cdots & 0 & z_j \end{pmatrix}.$$

Proof. Since $R_+ : \mathcal{H} \to L^2(\mathbb{R})$ is unitary, $R^{\pm}\mathcal{H}^{\pm} = L^2(\mathbb{R}_{\pm})$ and $R^{+}(R^{-})^{-1} = S_p^*$, we have

$$R^{+}\mathcal{K} = L^2(\mathbb{R}) \ominus ((L^2(\mathbb{R}_-) \cap S_p^* L^2(\mathbb{R}_-)) \oplus L^2(\mathbb{R}_+)).$$

By our hypotheses $p(\mathrm{i}\zeta) \neq 0$ for any ζ in the upper complex plane $\mathbb{C}_+$. Thus by Paley–Wiener theorems, the analytic extension to $\mathbb{C}_+$ of the Fourier transform of $f \in \mathcal{S}_0^- \cap S_p^* \mathcal{S}_0^-$,

$$\hat{f}(\zeta) = -\frac{p(-\mathrm{i}\zeta)}{p(\mathrm{i}\zeta)}\hat{g}(\zeta),$$

has no poles and has zeros of order ν_j at $\mathrm{i}z_j$, i.e.,

$$\left.\frac{\mathrm{d}^k \hat{f}(\zeta)}{\mathrm{d}\zeta^k}\right|_{\zeta = iz_j} = \frac{(-\mathrm{i})^k}{\sqrt{2\pi}} \int_{-\infty}^{0} x^k\, \mathrm{e}^{z_j x} f(x)\, \mathrm{d}x = 0, \qquad k = 0, 1, \ldots, \nu_j - 1.$$

Thus,

$$L^2(\mathbb{R}_-) \cap S_p^* L^2(\mathbb{R}_-) = \{x^k\, \mathrm{e}^{z_j x}, k = 0, 1, \ldots, \nu_j - 1, j = 1, \ldots, m\}^{\perp}$$

and the finite-dimensional subspace $\mathcal{K}$ is generated by vectors (7.1). These vectors are independent and so the dimension of $\mathcal{K}$ is $\sum_{j=1}^{m} \nu_j = \deg p$. To determine the action of Z^t on $\mathcal{K}$ it is enough to note that the evolution of vectors (7.1) in the outgoing representation is given by

$$\phi_{kj}(x - t) \equiv (x - t)^k\, \mathrm{e}^{z_j(x-t)} \chi_{(-\infty,0]}(x - t).$$

Thus,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\phi_{kj}(x - t)\right|_{t=0} = -(k\phi_{k-1,j}(x) + z_j\phi_{kj}(x))$$

and the proof is completed. □

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